Skolem Functions for Factored Formulas

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Skolem functions and their applications

CEGAR for Skolem functions

Experimental Results

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Definition (Skolem functions)

Given a propositional function $F(x, Y)$, a Skolem function for $x \in X$ in $F(x, Y)$ is a function $\psi(Y)$ such that

$$\exists x \; F \equiv F[x \mapsto \psi].$$

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3. Are Skolem functions unique?

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Definition (Skolem function vector)

Given a propositional function $F(X, Y)$, a Skolem function vector for $X = (x_1, \ldots x_n)$ in F is a vector of functions $\mathbf{\Psi} = (\psi_1, \ldots, \psi_n)$ such that

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Algorithm 1. $\textsc{SkolemGeneration}(F(x_1, \ldots, x_n, Y))$.

1. Input: Propositional formula $F(x_1, x_2, \ldots, x_n, Y)$
2. Output: Skolem function set $\mathbf{\Psi} = \{\psi_1, \ldots, \psi_n\}$
3. For $i = 1$ to n
   3.1 $\psi_i = \textsc{SkolemFun}(F, x_i)$
   3.2 $F = \exists x_i F = F[x_i \mapsto \psi_i]$
4. Return $\{\psi_1, \psi_2, \ldots, \psi_n\}$.

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Example 2. Find a Skolem function vector (ψ_1, ψ_2) for (x_1, x_2) in formula $F(x_1, x_2, y_1, y_2) = x_1 \wedge \overline{x_2} \wedge y_1 \wedge y_2$.

Our focus and applications

Skolem Generation for Factored Formulas

1. Given a propositional function $F(X, Y)$ as conjunction of factors,

$$f^1(X_1, Y_1) \wedge f^2(X_2, Y_2) \cdots \wedge f^r(X_r, Y_r)$$

find Skolem functions for variables in X.

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Skolem function generation is a key verification/synthesis problem due to its applications in:

1. Quantifier elimination, of course
2. Generating certificates in Quantified Boolean Formula (QBF) solving
3. Program synthesis
   - Combinatorial Sketching for Finite Programs [SLTB$^+$06]
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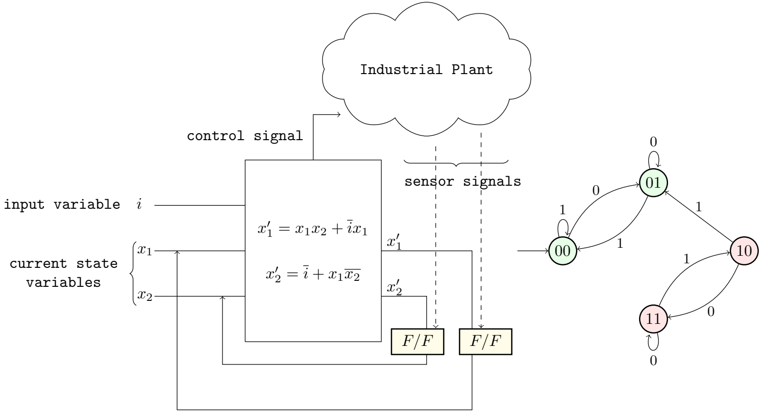
Plant Control Problem



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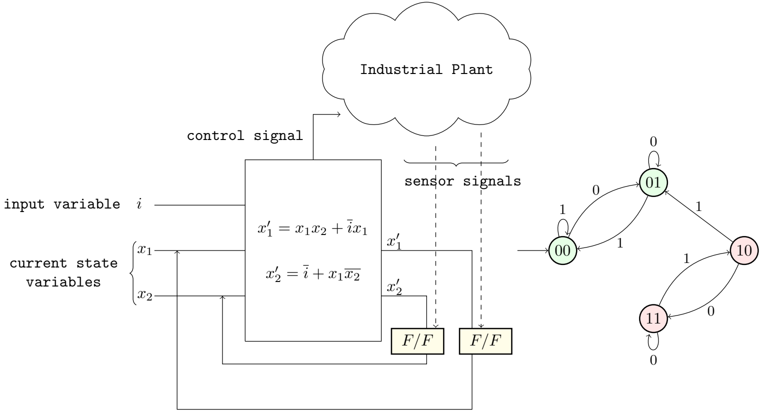
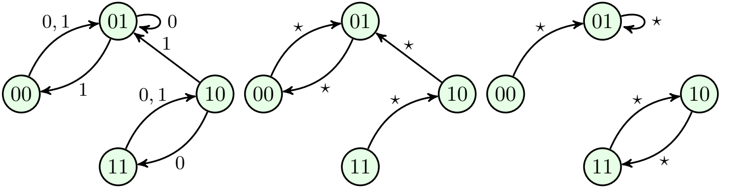


1. How should the primary input of the controller be driven so that the controller transitions to a desirable state in one step, whenever possible?
2. Solution: Find Skolem function for the primary input variable:

$$\exists x'_1 \exists x'_2 \left((x'_1 = x_1x_2 + \bar{i}x_1) \wedge (x'_2 = \bar{i} + x_1\overline{x_2}) \wedge \mathsf{Good}(x'_1, x'_2)\right)$$

Graph Decomposition Problem

$\textsc{NotCovered} = 1$ $\quad$ $\textsc{NotCovered} = \begin{array}{l}(\neg y_1 \wedge y_2 \wedge \neg x)\\ \vee (y_1 \wedge \neg y_2 \wedge \neg x)\end{array}$



$y_1' = (\neg x \wedge y_1) \vee (y_1 \wedge y_2)$
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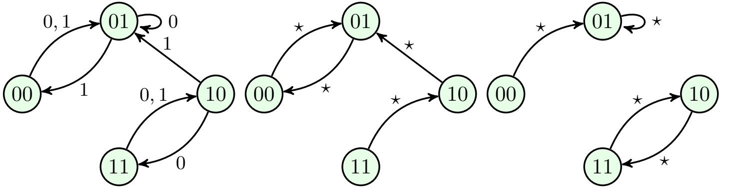

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1. Compute a disjunctive decomposition of implicitly specified state transition graphs of sequential circuits [Tri03, TCP08].
2. Solution: Find Skolem function for the input variables X in:

$$\wedge_i \textsc{NotCovered}_i(X, Y).$$

Existing Approaches

1. Extract Skolem function from the proof of validity of $\exists X F(X, Y)$
   - succinct Skolem functions if there exists a short proof of validity.

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3. Composition based approaches
   - Quantifier elimination via functional composition by Jiang [Jia09]
   - Techniques in Symbolic model checking by Trivedi [Tri03]
   - Work well for small-sized formulas
   - Compositions cause formula blow up and memory out

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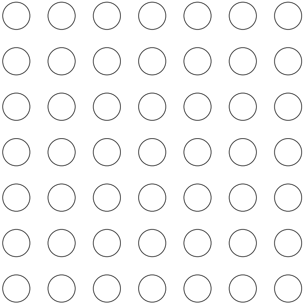
Experimental Results

Find $\psi(Y)$ such that $\exists x F(x, Y) \equiv F(\psi(Y), Y)$.



— Set of All valuations to Y.

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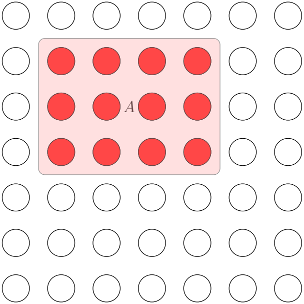

— $A(Y) =$ Can't set x to 1 to satisfy $F = \neg F(x, Y)[x \mapsto 1]$

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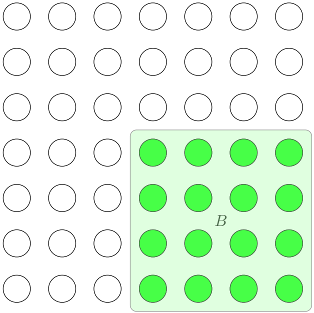

— $B(Y) =$ Can't set x to 0 to satisfy $F = \neg F(x, Y)[x \mapsto 0]$

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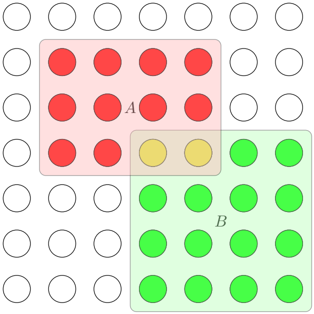

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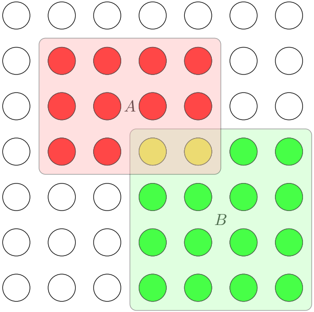

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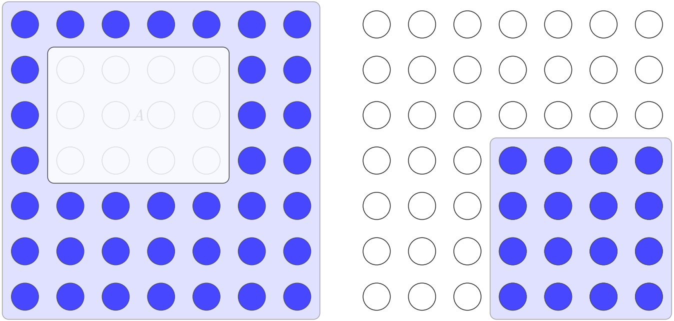

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- A Skolem function for x in F is any Interpolant of $(B \setminus A)$ and $(A \setminus B)$
- E.g. $\neg A = F(x, Y)[x \mapsto 1] = F(1, Y)$
- and $B = \neg F(x, Y)[x \mapsto 0] = \neg F(0, Y)$.

Monolithic Skolem Generation [Jia09, Tri03]

	f^1	f^2	f^3	$\cdots$	f^k	$\cdots$	f^r	ψ_i	$\exists x_i F$
x_1	$\boxed{\cdot}$		$\boxed{\cdot}$						
x_2		$\boxed{\cdot}$	$\boxed{\cdot}$						
$\vdots$									
x_ℓ	$\boxed{\cdot}$	$\boxed{\cdot}$			$\boxed{\cdot}$		$\boxed{\cdot}$		
$\vdots$									
x_n	$\boxed{\cdot}$						$\boxed{\cdot}$		
Y									

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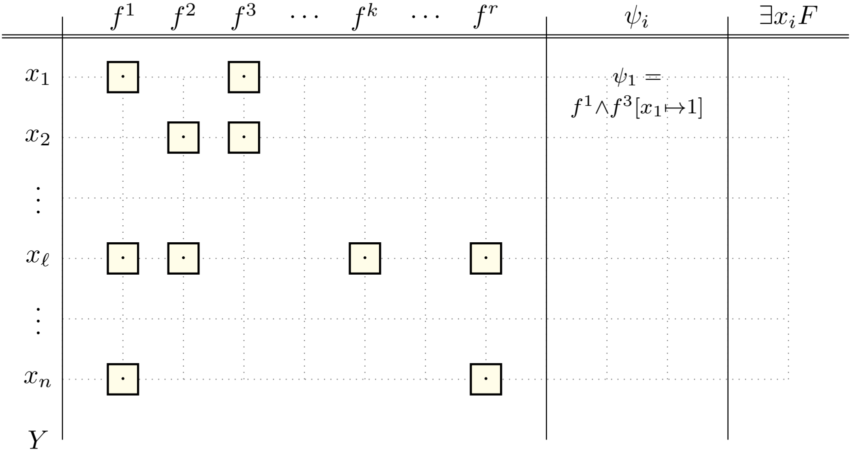

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x_1	$\boxed{\cdot}$		$\boxed{\cdot}$					$\psi_1 =$ $f^1 \wedge f^3[x_1 \mapsto 1]$	$f^1 = f^1[x_1 \mapsto \psi_1]$ $f^3 = f^3[x_1 \mapsto \psi_1]$
x_2		$\boxed{\cdot}$	$\boxed{\cdot}$						
$\vdots$									
x_ℓ	$\boxed{\cdot}$	$\boxed{\cdot}$			$\boxed{\cdot}$		$\boxed{\cdot}$		
$\vdots$									
x_n	$\boxed{\cdot}$						$\boxed{\cdot}$		
Y									

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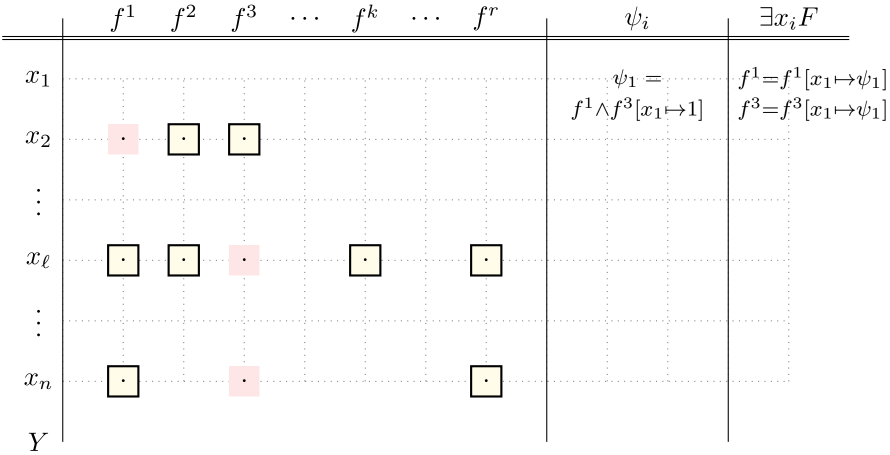

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x_1								$\psi_1 = f^1 \wedge f^3[x_1 \mapsto 1]$	$f^1 = f^1[x_1 \mapsto \psi_1]$ $f^3 = f^3[x_1 \mapsto \psi_1]$
x_2	$\cdot$	$\boxed{\cdot}$	$\boxed{\cdot}$					$\psi_2 = f^1 \wedge f^2 \wedge f^3[x_2 \mapsto 1]$	
$\vdots$									
x_ℓ	$\boxed{\cdot}$	$\boxed{\cdot}$	$\cdot$		$\boxed{\cdot}$		$\boxed{\cdot}$		
$\vdots$									
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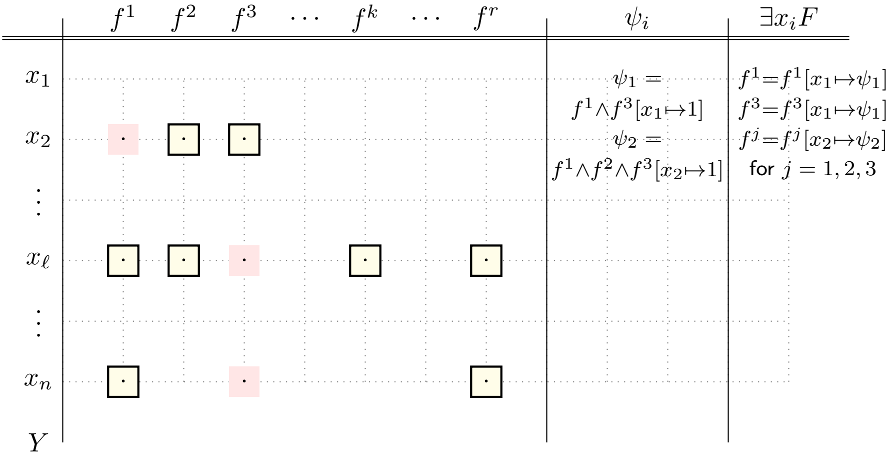

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x_2								$\psi_2 = f^1 \wedge f^2 \wedge f^3[x_2 \mapsto 1]$	$f^j = f^j[x_2 \mapsto \psi_2]$ for $j = 1, 2, 3$
$\vdots$									
x_ℓ	$\boxed{\cdot}$	$\boxed{\cdot}$	$\cdot$		$\boxed{\cdot}$		$\boxed{\cdot}$		
$\vdots$									
x_n	$\boxed{\cdot}$	$\cdot$	$\cdot$				$\boxed{\cdot}$		
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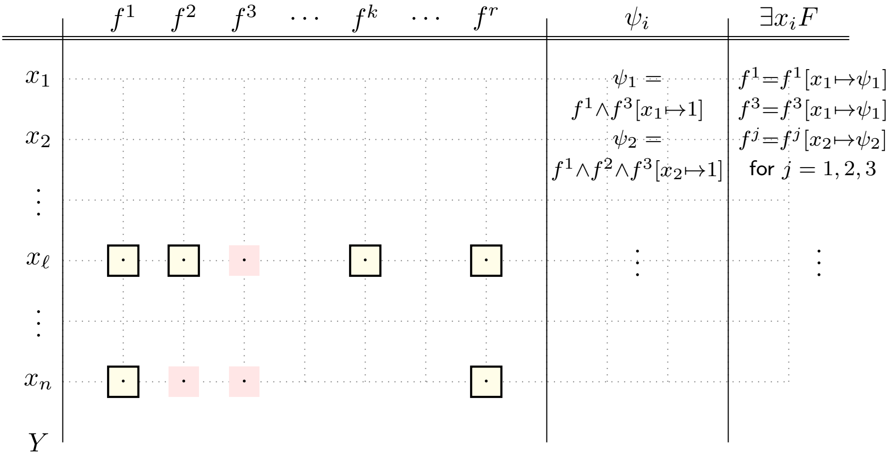

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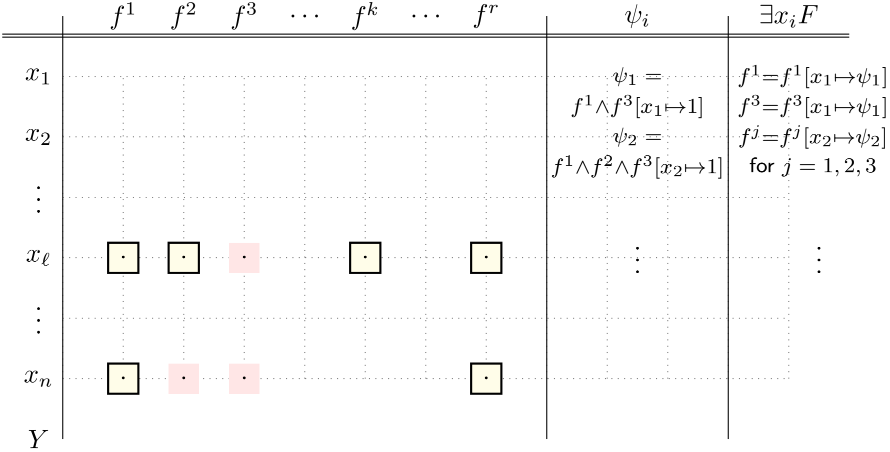
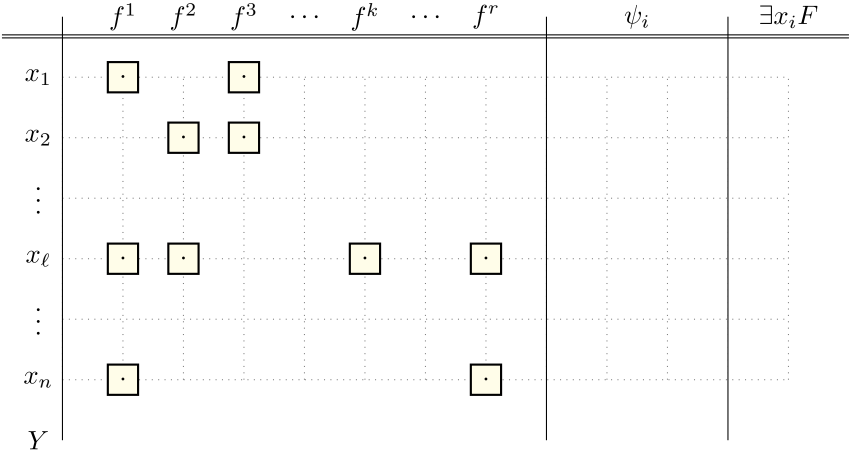


Blowup in sizes of factors after each quantifier elimination

How to avoid such blowup



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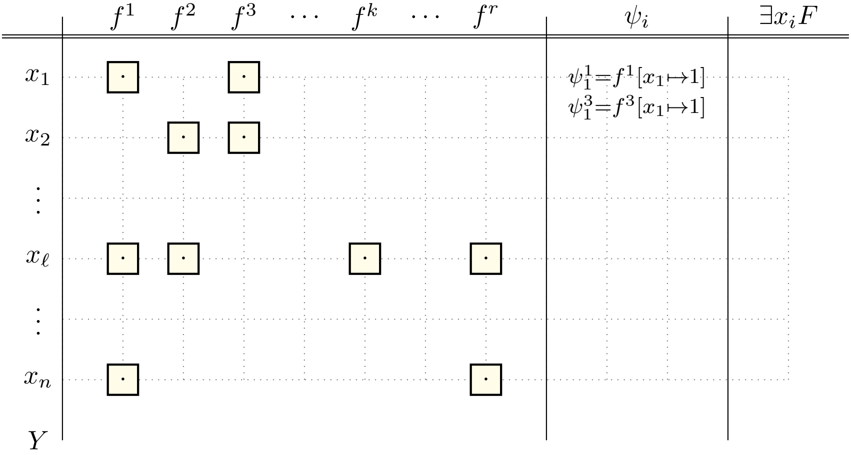

How to avoid such blowup

	f^1	f^2	f^3	$\dots$	f^k	$\dots$	f^r	ψ_i	$\exists x_i F$
x_1	$\boxed{\cdot}$		$\boxed{\cdot}$					$\psi_1^1 = f^1[x_1 \mapsto 1]$ $\psi_1^3 = f^3[x_1 \mapsto 1]$	$f^1 = f^1[x_1 \mapsto \psi_1^1]$ $f^3 = f^3[x_1 \mapsto \psi_1^3]$
x_2		$\boxed{\cdot}$	$\boxed{\cdot}$						
$\vdots$									
x_ℓ	$\boxed{\cdot}$	$\boxed{\cdot}$			$\boxed{\cdot}$		$\boxed{\cdot}$		
$\vdots$									
x_n	$\boxed{\cdot}$						$\boxed{\cdot}$		
Y									

How to avoid such blowup

	f^1	f^2	f^3	$\dots$	f^k	$\dots$	f^r	ψ_i	$\exists x_i F$
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x_2		$\boxed{\cdot}$	$\boxed{\cdot}$						
$\vdots$									
x_ℓ	$\boxed{\cdot}$	$\boxed{\cdot}$			$\boxed{\cdot}$		$\boxed{\cdot}$		
$\vdots$									
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x_2		$\boxed{\cdot}$	$\boxed{\cdot}$					$\psi_2^2 = f^2[x_2 \mapsto 1]$ $\psi_2^3 = f^3[x_2 \mapsto 1]$	
$\vdots$									
x_ℓ	$\boxed{\cdot}$	$\boxed{\cdot}$			$\boxed{\cdot}$		$\boxed{\cdot}$		
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$\vdots$									
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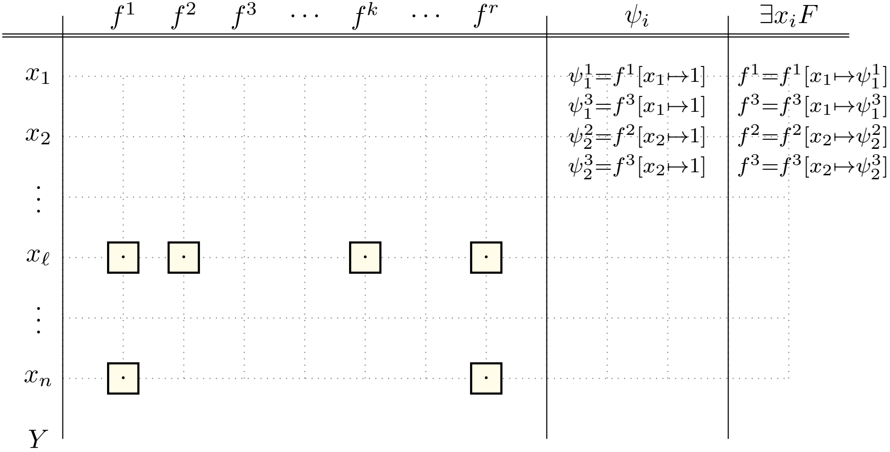

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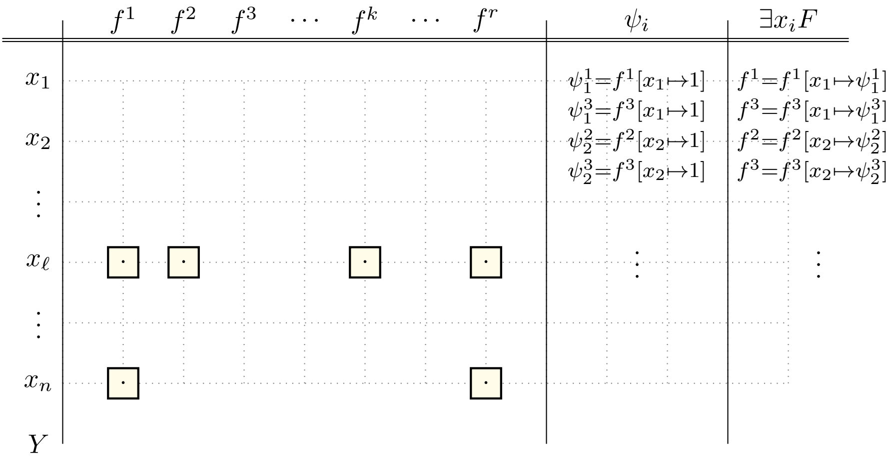
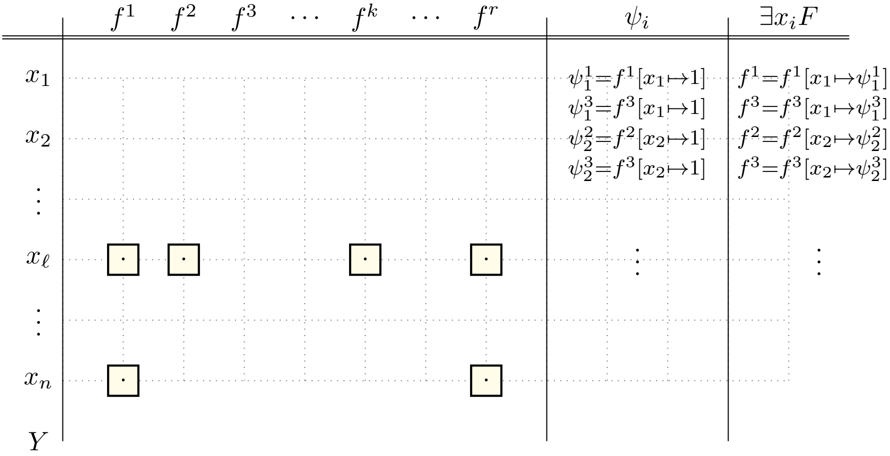


— Skolem functions are in factored form: $\psi_i = \wedge \psi_i^k$

How to avoid such blowup



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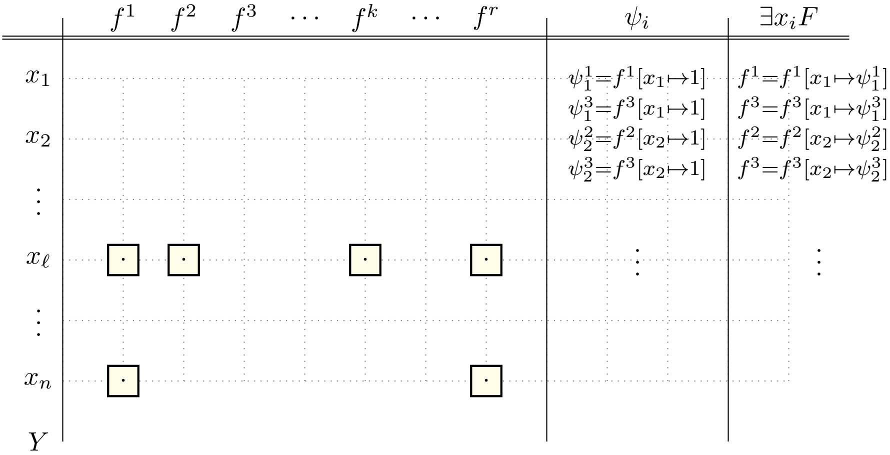

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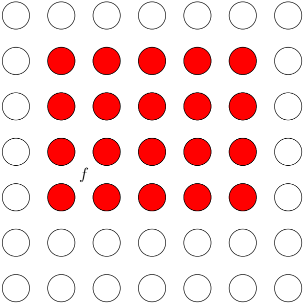
— Abstraction of $\exists x_i F$ and of ψ_i

Counterexample-Guided Abstraction Refinement

– Given propositional functions $f(X)$ and $g(X)$, we say that f is an abstraction of g if

$$g \implies f$$



– We will also say that g is a refinement of g.

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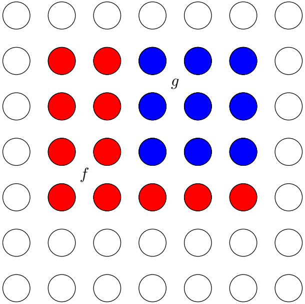

- We will also say that g is a refinement of g.

CEGAR: Contd

- An abstract Skolem function is a function that is an abstraction of a proper Skolem function.
- An abstraction Skolem function may not be a proper Skolem function.
- Given a formula $F(x_1, \ldots, x_n, Y)$ and functions $\Psi = \{\psi_1, \psi_2, \ldots, \psi_n\}$ how do we check if Ψ is a proper Skolem vector?

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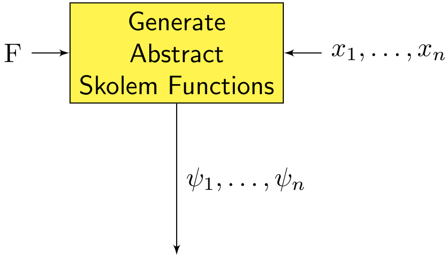
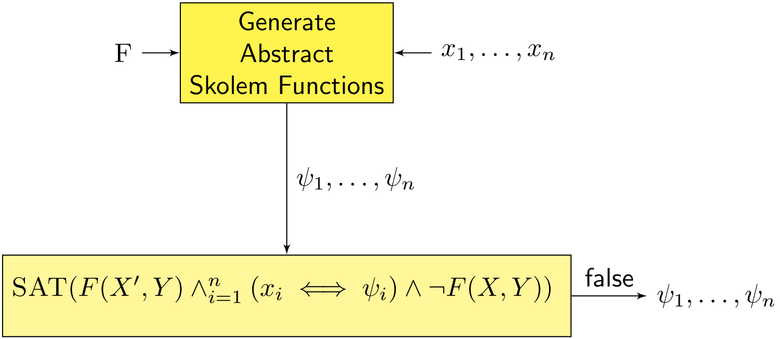
- If this formula is unsatisfiable, then $\psi_1, \ldots, \psi_n$ are proper Skolem functions for $x_1, \ldots, x_n$
- Otherwise, satisfying assignment helps us to refine Skolem function.

CEGAR: Contd.

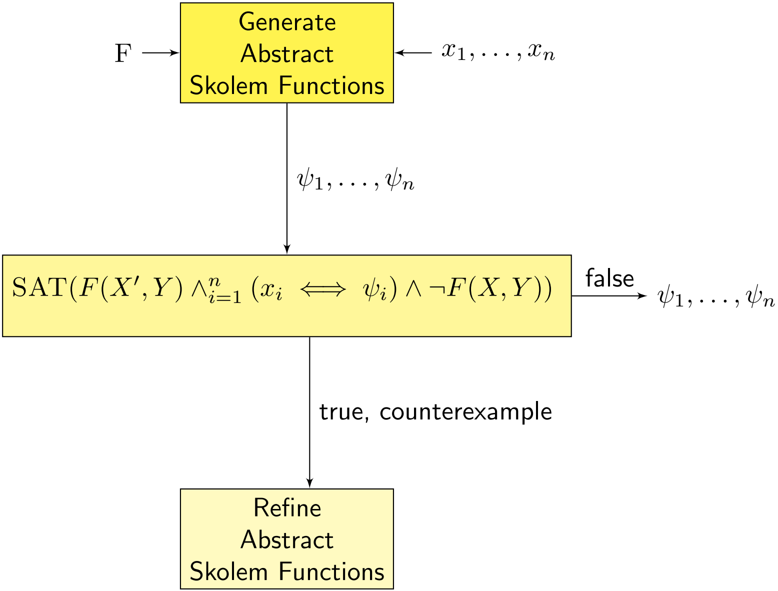


F
Generate Abstract Skolem Functions
$x_1, \ldots, x_n$
$\psi_1, \ldots, \psi_n$

CEGAR: Contd.


F
Generate
Abstract
Skolem Functions
$x_1, \dots, x_n$
$\psi_1, \dots, \psi_n$
$\mathrm{SAT}(F(X', Y) \wedge_{i=1}^{n} (x_i \iff \psi_i) \wedge \neg F(X, Y))$
false
$\psi_1, \dots, \psi_n$

CEGAR: Contd.


Generate Abstract Skolem Functions
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$x_1, \ldots, x_n$
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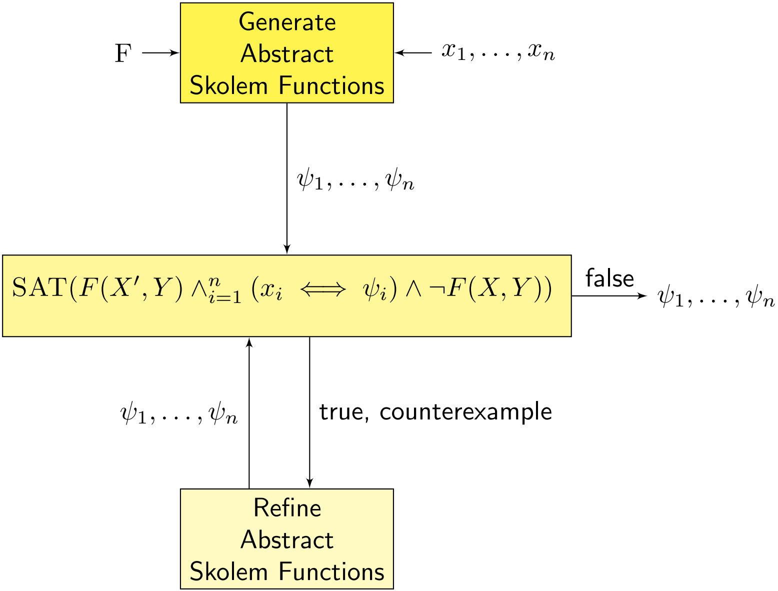
false
$\psi_1, \ldots, \psi_n$
true, counterexample
Refine Abstract Skolem Functions

CEGAR: Contd.



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Generate Abstract Skolem Functions
$x_1, \ldots, x_n$
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false
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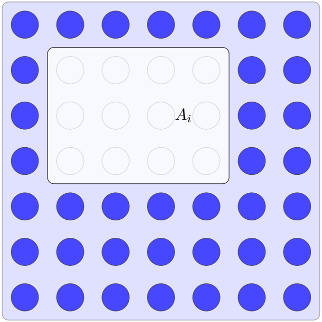
Refine Abstract Skolem Functions

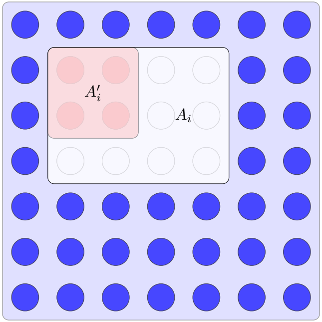
Abstraction and Refinement



$A_i = \neg\exists x_1 \ldots x_{i-1} F[x_i \mapsto 1]$ and $\psi_i = \neg A_i$

1. Ideally when we need to compute Skolem function for x_i we need to have access to $F_i = \exists x_1, \ldots x_{i-1} F$.
2. Then, to compute Skolem function we can compute the set $A_i = \neg F_i[x \mapsto 1]$ and a proper Skolem function would be $\neg A$.

Abstraction and Refinement



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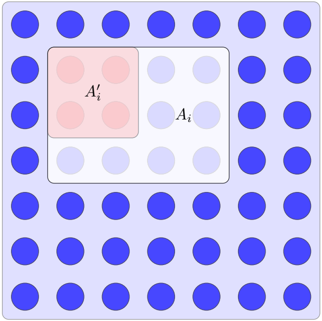
4. Hence, the set A_i' computed using F_i' would be a refinement of the proper A_i.

Abstraction and Refinement



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$A_i' = \neg F_i'[x_i \mapsto 1]$ and $\psi_i' = \neg A_i'$

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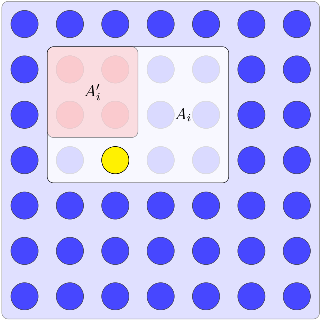
3. However, due to factorwise quantification, we only know an abstraction F_i' of F_i.
4. Hence, the set A_i' computed using F_i' would be a refinement of the proper A_i.
5. This implies that the Skolem function computed as $\neg A_i'$ will be an abstract Skolem function.

Abstraction and Refinement



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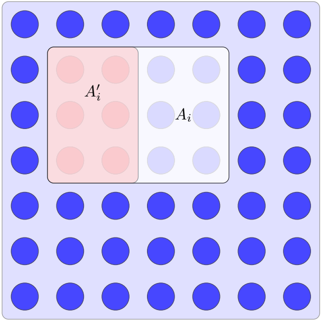
Abstraction and Refinement



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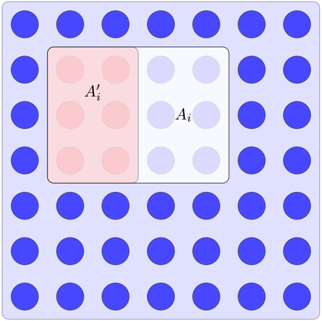

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3. CEGAR loop continues in this way until we find proper Skolem functions.

Skolem functions and their applications

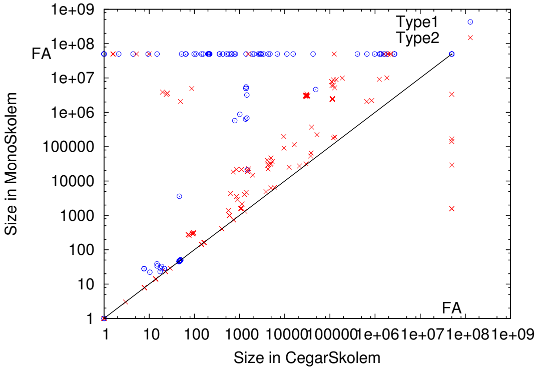
CEGAR for Skolem functions

Experimental Results

Benchmarks

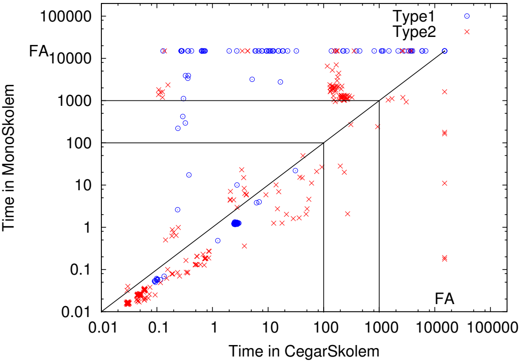
1. We compared the performance of the CEGAR based algorithm with
   1.1 an implementation of the monolithic algorithm
   1.2 The tool Bloqqer (a QRAT based Skolem function generation tool).
2. Our benchmarks were obtained by considering the disjunctive decomposition problem for sequential circuits from HWMCC10 benchmark suite
3. We divided our benchmarks into TYPE-1 formula where $\exists X F(X, Y)$ is valid (160 benchmarks) and TYPE-2 formulas where $\exists X F(X, Y)$ is not valid (264 benchmarks).
4. We used ABC library to represent and manipulate functions as AIGs and used default SAT solver provided by ABC (a variant of miniSAT).
5. We compared these algorithms with respect to Skolem function size and total time taken to generate Skolem functions
6. The maximum time and memory usage was restricted to 2 hours and 32GB.

Monolithic Vs CEGAR: Size



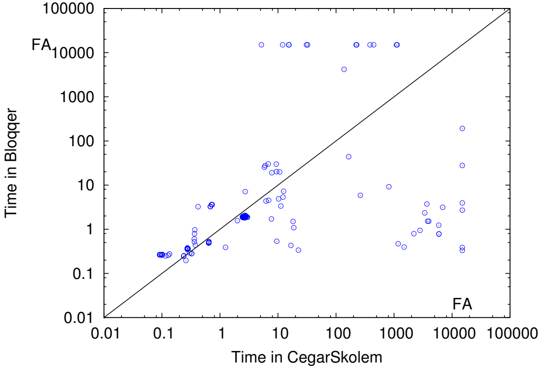
1. There is no instance on which CEGAR generates Skolem functions that are larger on average than Monolithic.

Monolithic Vs CEGAR: Time



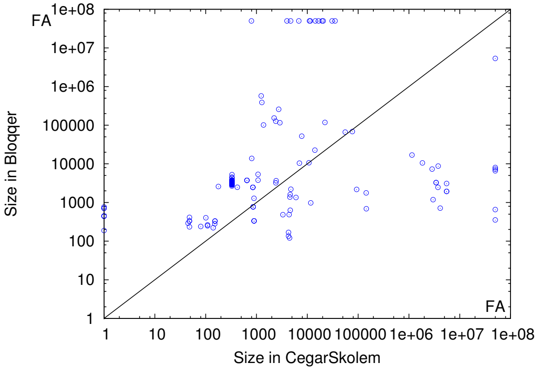
1. Due to repeated calls to SAT solver, CEGAR took more time than Monolithic, but for those examples total time in < 100 seconds.
2. For timed between 100 and 300, Monolithic performed much worse taking more than 1000 seconds (due to large sizes of Skolem functions)
3. Monolithic timed out for 83 benchmarks, while CEGAR for 10

Bloqqer Vs CEGAR: Time



1. Out of 160 TYPE-1 benchmarks Bloqqer generated Skolem functions for 148 benchmarks and gave NOT_VERIFIED message for the remaining.
2. CEGAR was successful for 154 benchmarks.
3. For the benchmarks where Bloqqer gave NOT VERIFIED message, 8 of these 12 were large benchmarks with $1000+$ factors and variables

Bloqqer Vs CEGAR: Size



1. For the 142 common benchmarks, in majority of the cases $(108/142)$ CEGAR generated smaller Skolem functions.

Conclusion

1. Presented a Counterexample guided abstraction refinement based algorithm to generate Skolem functions for factored propositional formulas
2. Experiments show that for complex functions, our algorithm significantly outperformed two state-of-the-art algorithms
3. As a future work, we plan to explore integration with more efficient SAT-solvers, and refinement using multiple counter-examples in parallel.

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Thank you

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