ACL2(r) Formalization of Fourier Series' Properties

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Motivation

Fourier series have many applications to a wide variety of mathematical and physical problems, electrical engineering, signal processing, etc.

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We are interested in formalizing Fourier series (and possibly, Fourier transform) in ACL2 as a useful tool for formally analyzing analog circuits, mixed-signal integrated circuits, etc.

Overview

We formalize some basic properties of Fourier series in the logic of ACL2(r), which is a variant of ACL2 that supports reasoning about the real numbers by way of non-standard analysis [R. Gamboa, 1999].

Non-Standard Analysis

Formulate the operations of calculus using a logically rigorous notion of infinitesimal numbers, instead of epsilon-delta definition of limit.

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1. Extend the reals to a bigger set of hyperreals, which includes infinitesimals [A. Robinson, 1996].
2. Nelson's Internal Set Theory views the "reals" as "all the reals", including infinitesimals, and considers a subset of standard reals [E. Nelson, 1977].

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Why use non-standard analysis in ACL2?

- ACL2 has very limited support for reasoning with quantifiers.

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Fourier coefficient formulas

Sum rule for integration of infinite series

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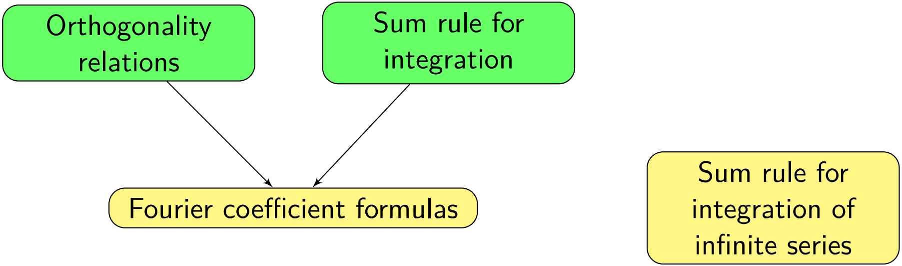

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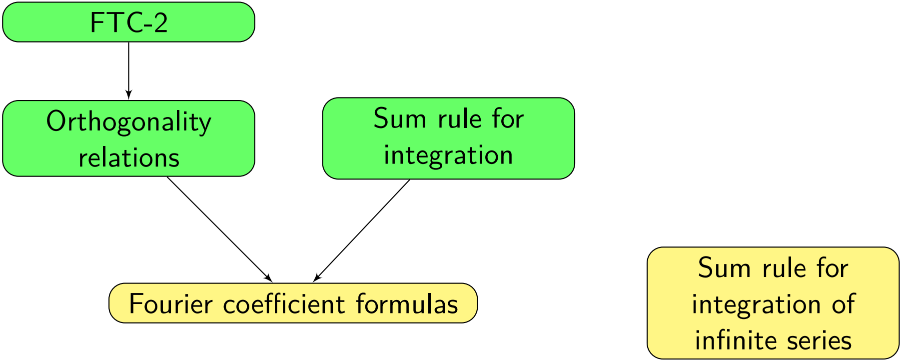

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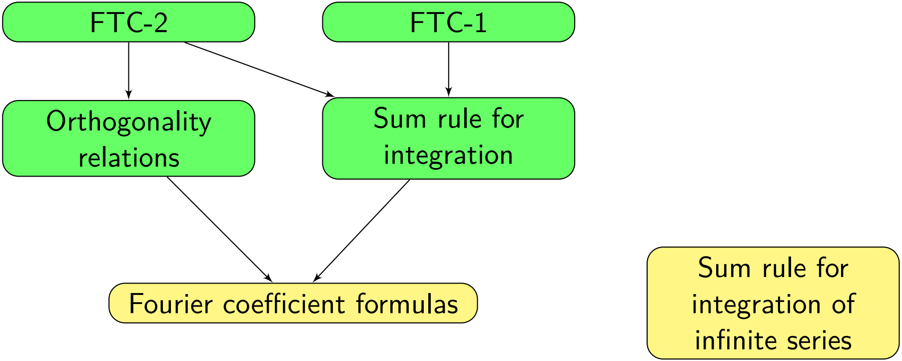

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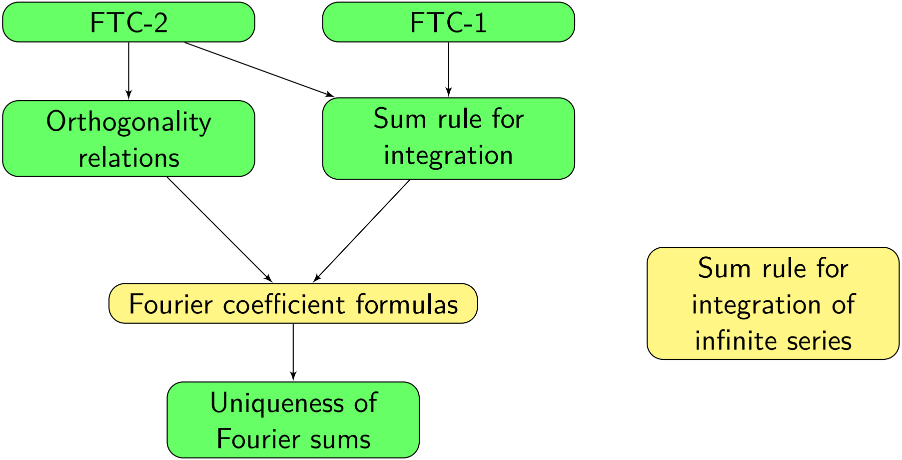

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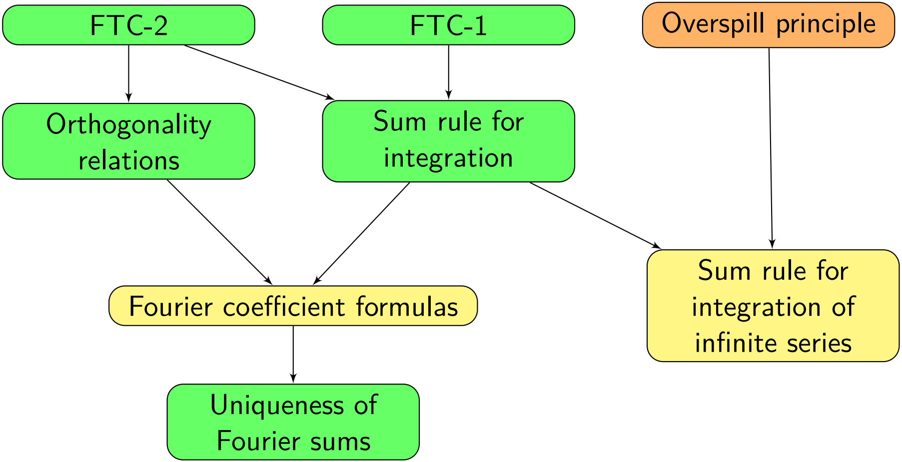

Fourier Coefficient Formulas

Theorem 1 (Fourier coefficient formulas)

Consider the following Fourier sum for a periodic function with period $2L$:

$$f(x) = a_0 + \sum_{n=1}^{N} \left(a_n \cos(n\tfrac{\pi}{L}x) + b_n \sin(n\tfrac{\pi}{L}x)\right)$$

Then

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\frac{\pi}{L}x)dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\frac{\pi}{L}x)dx.$$

Sum Rule for Definite Integrals of Infinite Series

The result presented in the previous slide just applies to finite sums. However, Fourier series can be infinite.

Formalizing the sum rule for definite integrals of infinite series under certain conditions.

$$\int_a^b \lim_{N \to \infty} \left(\sum_{n=0}^{N} f_n(x) \right) dx \stackrel{?}{=} \lim_{N \to \infty} \left(\sum_{n=0}^{N} \int_a^b f_n(x) dx \right)$$

Thank You!

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