Compact Symmetry Breaking for Tournaments

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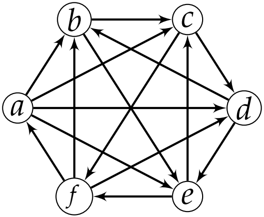
Tournaments

- Complete, Directed Graphs
- Goal: Solve Tournament Existence Problems
- Labeled vs Unlabeled



	a	b	c	d	e	f
a	0	1	1	1	1	0
b	0	0	1	0	1	0
c	0	0	0	1	0	1
d	0	1	0	0	1	0
e	0	0	1	0	0	1
f	1	1	0	1	0	0

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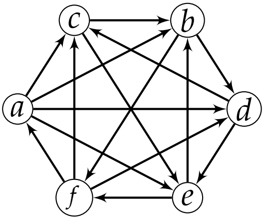

	a	b	c	d	e	f
a	0	1	1	1	1	0
b	0	0	0	1	0	1
c	0	1	0	0	1	0
d	0	0	1	0	1	0
e	0	1	0	0	0	1
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Why Search for Tournaments?

Doubly-Regular Tournaments $\sim$ Skew Hadamard Matrices
[Reid & Brown, 1972]

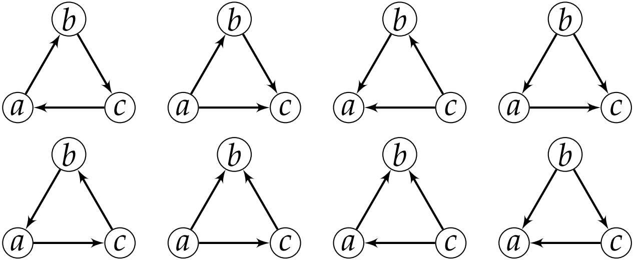
Hadamard Matrices have many applications:

- Feedback Delay Networks
- Compressed Sensing
- Quantum Computing

(Tournament) Ramsey Numbers

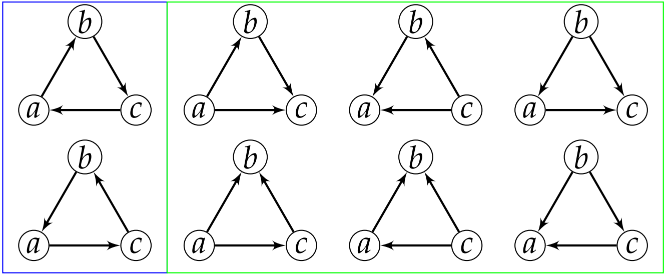
How to Search For Tournaments

Symmetry-Breaking Predicates for Labeled Tournaments



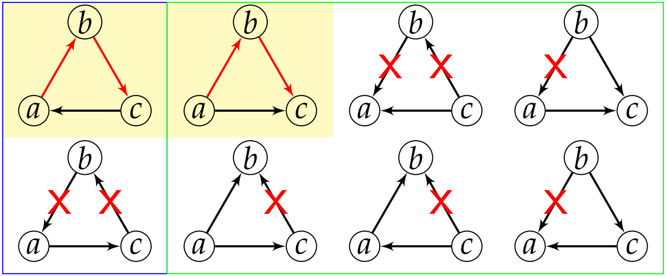
How to Search For Tournaments

Symmetry-Breaking Predicates for Labeled Tournaments



How to Search For Tournaments

Symmetry-Breaking Predicates for Labeled Tournaments



$$e_{ab} \wedge e_{bc} \quad \text{vs}$$

$$(e_{ab} \vee e_{bc} \vee e_{ac}) \wedge (e_{ab} \vee \overline{e_{bc}} \vee \overline{e_{ac}}) \wedge (e_{ab} \vee e_{bc} \vee \overline{e_{ac}}) \wedge$$
$$(\overline{e_{ab}} \vee e_{bc} \vee \overline{e_{ac}}) \wedge (\overline{e_{ab}} \vee e_{bc} \vee e_{ac}) \wedge (e_{ab} \vee \overline{e_{bc}} \vee e_{ac})$$

Isolator Properties

- A set of clauses is an Isolator if it admits at least one member of each equivalence class
- An isolator is Perfect if it admits exactly one such member
- A perfect isolator is Optimal if there is no other perfect isolator with fewer clauses

Method 1: Search With SAT

For any # of vertices n, # of clauses k in generated isolator:

- Calculate equivalence class of all labeled n-vertex tournaments
- Encode "Exactly one graph from each equivalence class satisfies the k-clause isolator"
- Break symmetries in encoding: clause ordering, vertex ordering, unit propagation

Method 1: Search With SAT

Properties:

- Isolator is *Perfect*
- Can determine *Optimal* isolator
- Likely intractable for $n > 6$

Method 2: Random Probes

For any # of vertices n:

- Calculate equivalence class of all labeled n-vertex tournaments
- Iteratively add clauses to reduce the number of redundant graphs

Method 2: Random Probes

Properties:

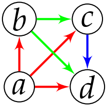
- Isolator is *Perfect*
- "Short", but likely no longer *Optimal*
- Intractable for $n > 8$ (exponential growth of number of labeled n-vertex tournaments)

Method 3: TT-fixing

Idea: tournament Ramsey numbers $R(k)$ guarantee that a transitive tournament of size k (a TT_k) is a sub-tournament of any tournament with $n \geq R(k)$ vertices.

We "fix" these tournaments in place via unit clauses.

Example (TT_4):



Method 3: TT-fixing

$$n = 16$$

$$R(5) = 14$$
$$R(4) = 8$$
$$R(3) = 4$$
$$R(2) = 2$$

$$\begin{bmatrix}
0&1&1&1&1&&&&&&&&&&& \\
0&0&1&1&1&&&&&&&&&&& \\
0&0&0&1&1&&&&&&&&&&& \\
0&0&0&0&1&&&&&&&&&&& \\
0&0&0&0&0&&&&&&&&&&& \\
&&&&&0&1&1&1&&&&&&& \\
&&&&&0&0&1&1&&&&&&& \\
&&&&&0&0&0&1&&&&&&& \\
&&&&&0&0&0&0&&&&&&& \\
&&&&&&&&&0&1&1&&&& \\
&&&&&&&&&0&0&1&&&& \\
&&&&&&&&&0&0&0&&&& \\
&&&&&&&&&&&&0&1&1& \\
&&&&&&&&&&&&0&0&1& \\
&&&&&&&&&&&&0&0&0& \\
&&&&&&&&&&&&&&&0
\end{bmatrix}$$

Method 3: TT-fixing

Properties:

- Isolator is neither *Perfect* nor *Optimal*
- Very short ($\Theta(n \log n)$ unit clauses) compared to exponential growth in unlabeled n-vertex tournaments
- Easily computable for all n via tournament Ramsey number upper bounds
- *Asymptotically Optimal*; best possible isolator within a constant factor as n grows large

Application to Improving Ramsey Theory

- Tournament Ramsey number $34 \leq R(7) \leq 47$ [Neiman, Mackey, Heule, 2020]
- To improve lower bounds on $R(k)$, find larger tournaments without TT_k
- 5303 previously known 33-vertex tournaments without TT_7

Application to Improving Ramsey Theory

Key idea: use SAT solver and small perfect isolator to find only new TT_7-free $n = 33$ tournaments

- Fix the TT_6-free 26-vertex subgraph in place
- Admit exactly one of each equivalence class for the remaining 7 vertices
- Disallow all cross-edges between the 26 and 7-vertex parts that match the 5303 currently known tournaments
- Resulted in 1 new TT_7-free $n = 33$ tournament

Conclusions

- Compact perfect isolators have been computed for $n \leq 8$
- TT-fixing generates asymptotically optimal isolators for arbitrary n
- Short, perfect isolators can be used to solve interesting tournament existence problems