BaxMC: a CEGAR approach to Max#SAT

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VERIMAG - UGA

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fmcad.22

Introduction

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In security: the question of *quantitative reachability*.

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atk = input()
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  win() # trigger / exploit vulnerability
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Useful applications:

- Comparing bugs quantitatively
- Measure the information leakage of a program

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Definition of the problem

Definition (Max#SAT)

Given $\phi(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$, find a witness x_m that maximizes the following:

$$|\phi(x_m, \mathcal{Y}, \mathcal{Z})| \triangleq |\{y \mid \exists z \,.\, \phi(x_m, y, z)\}|$$

Example

With $\mathcal{X} = \{a, b\}$, $\mathcal{Y} = \{c\}$, $\mathcal{Z} = \{d\}$:

$$\phi = (\neg a \vee c) \wedge (a \vee \neg b) \wedge (\neg b \vee d)$$

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A CEGAR-based algorithm

Inspired by CEGAR methods: the formula is refined towards the search of better solutions.

Key Insight

It is possible to quickly overapproximate the maximum model counting of a subspace.

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It is possible to quickly overapproximate the maximum model counting of a subspace.

If this overapproximation is below the current known maximum $\rightarrow$ we can exclude the sub-space from future search.

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Main algorithm

Takes $\phi(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ as input and returns x_m.

1. Find an x candidate
2. *Generalize* it to a partial witness $x|_{\mathcal{E}}$ such that $|\phi(x|_{\mathcal{E}}, \mathcal{Y}, \mathcal{Z})|$ is close (from below) to the current maximum
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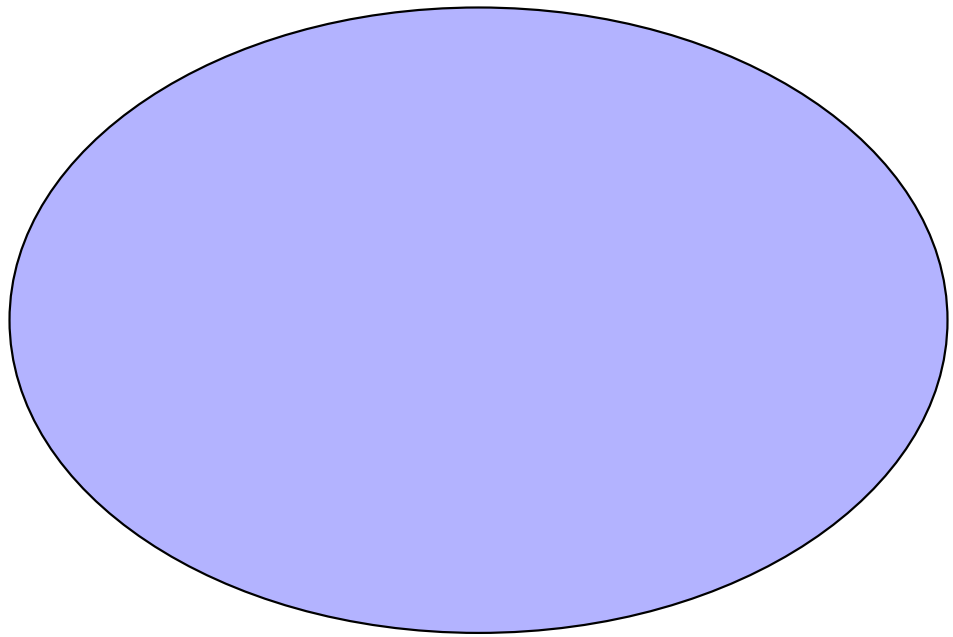

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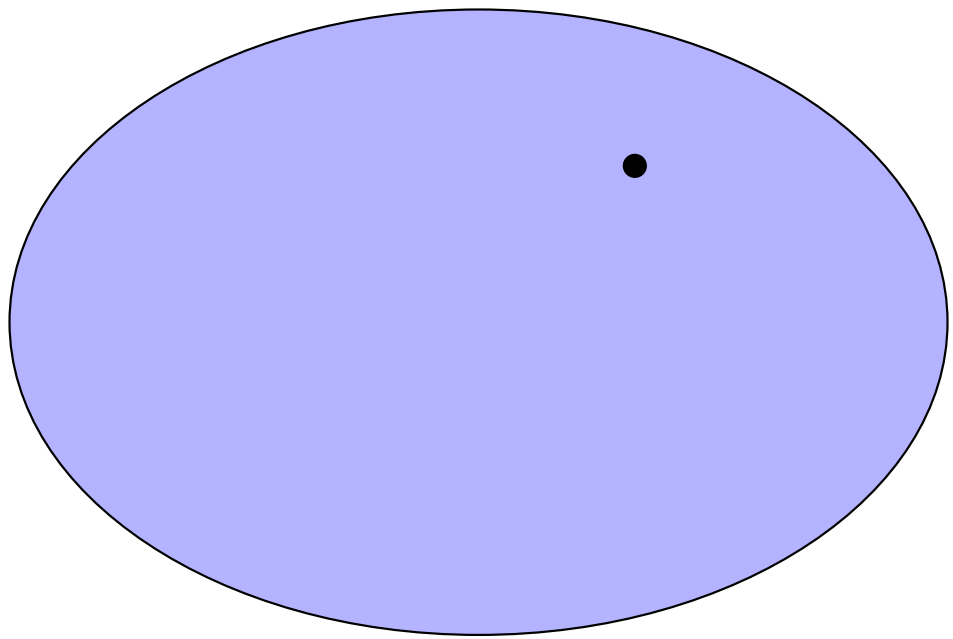

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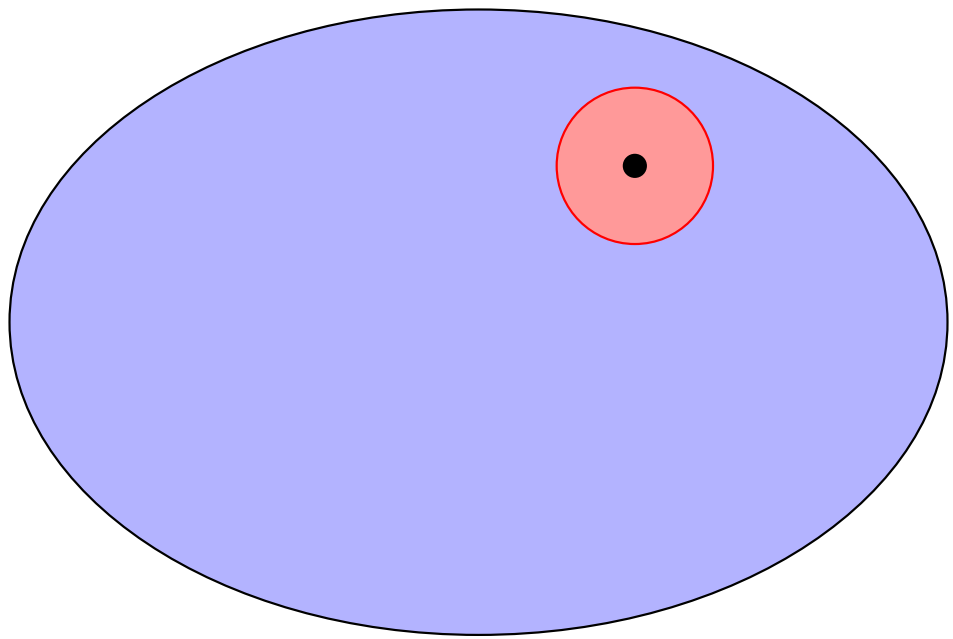

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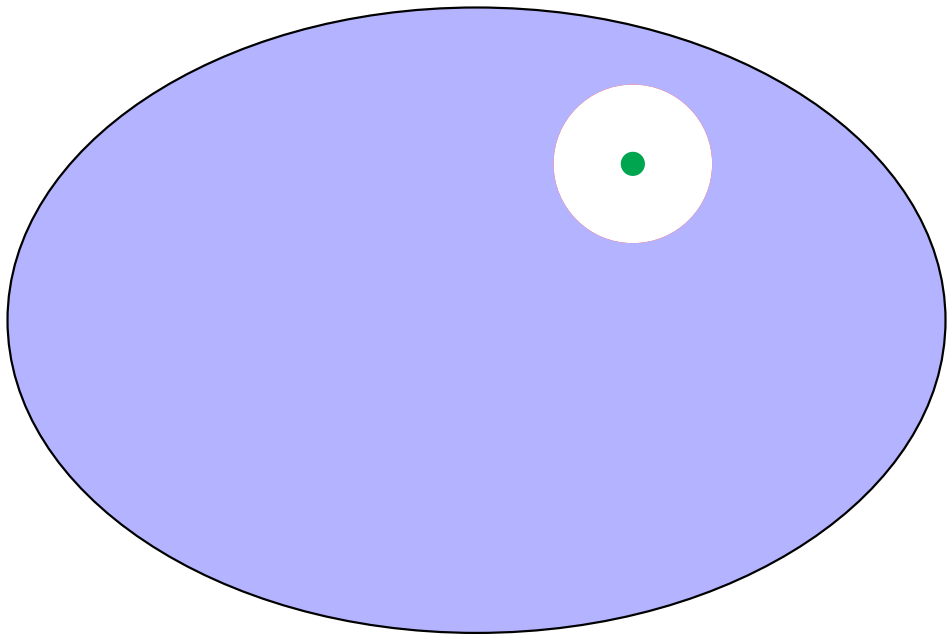

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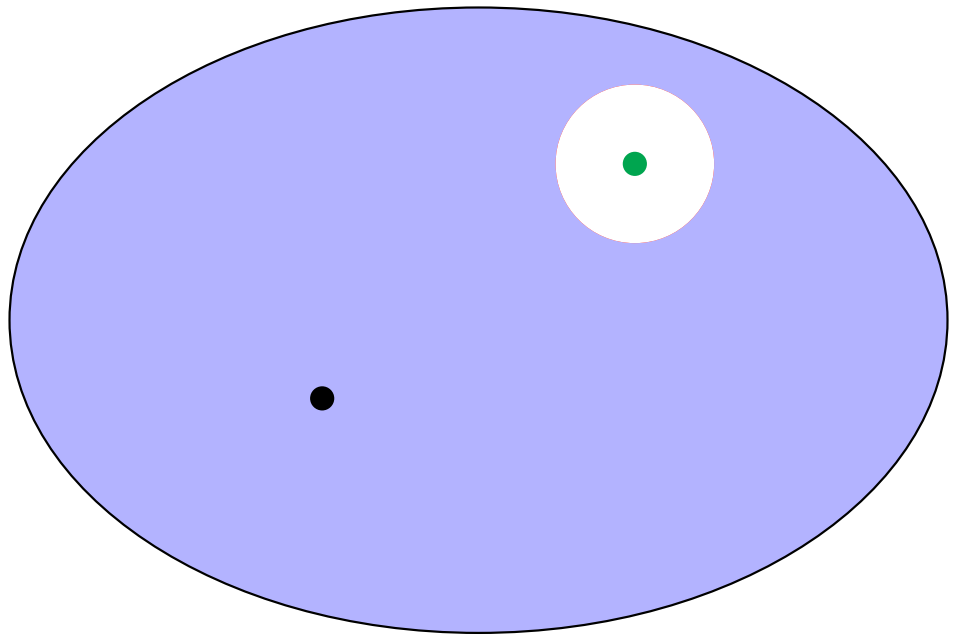

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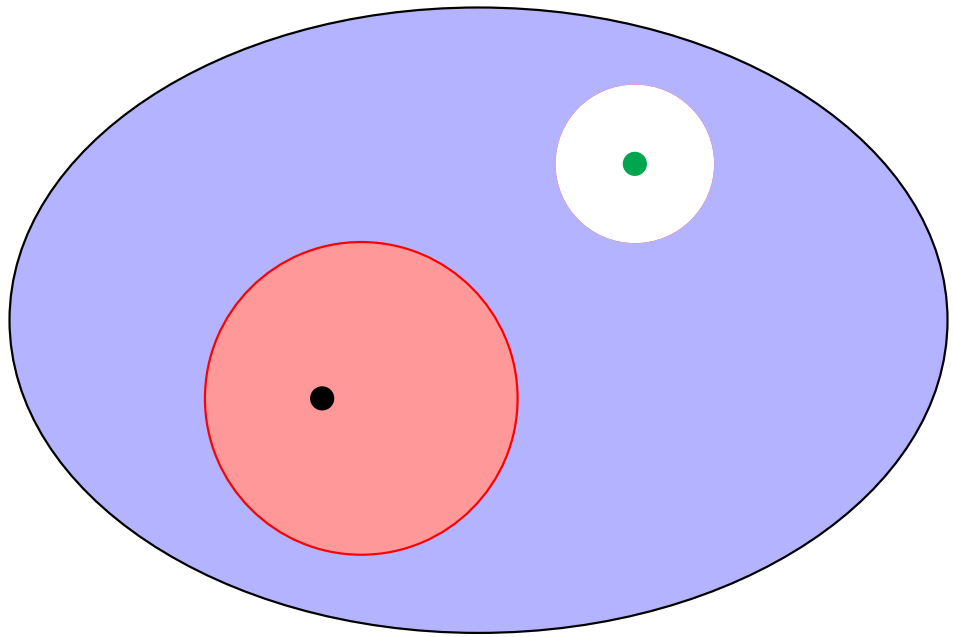

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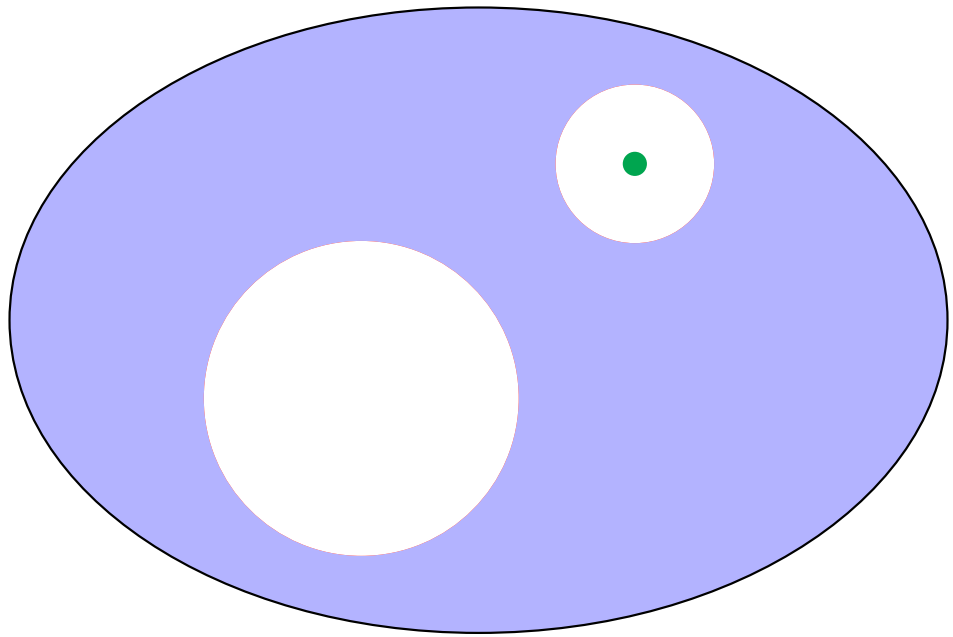

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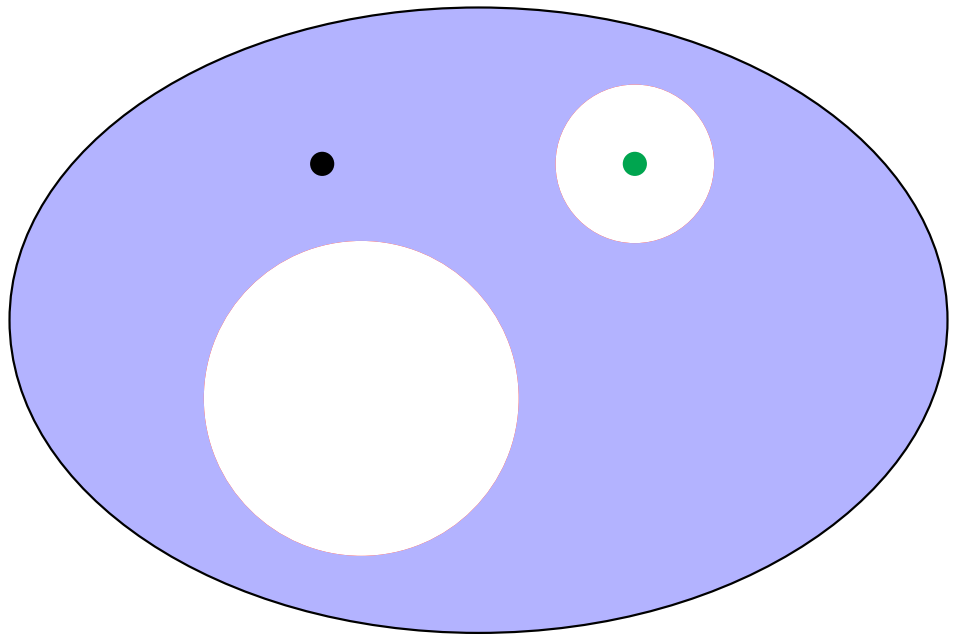

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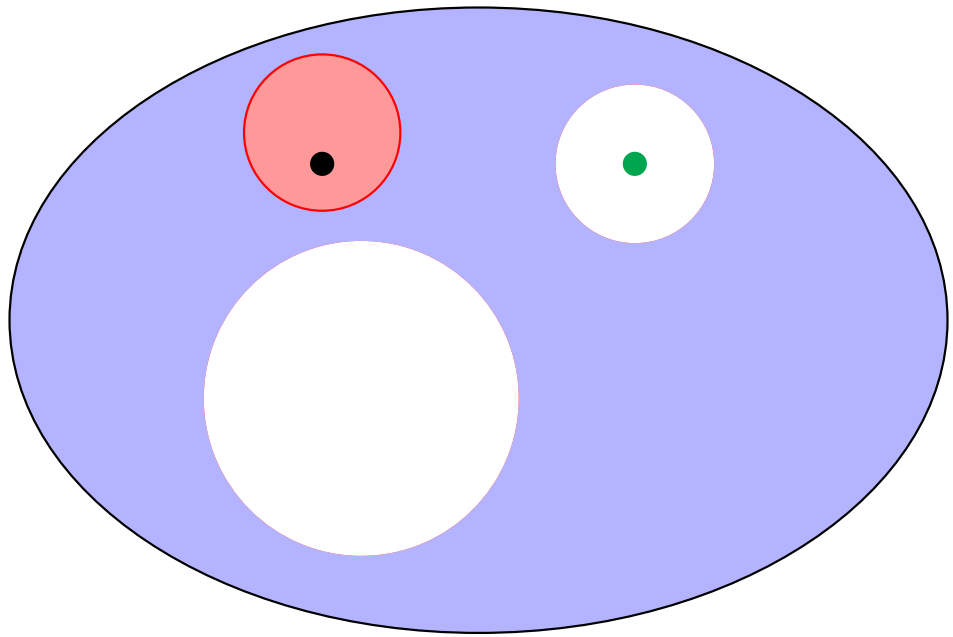

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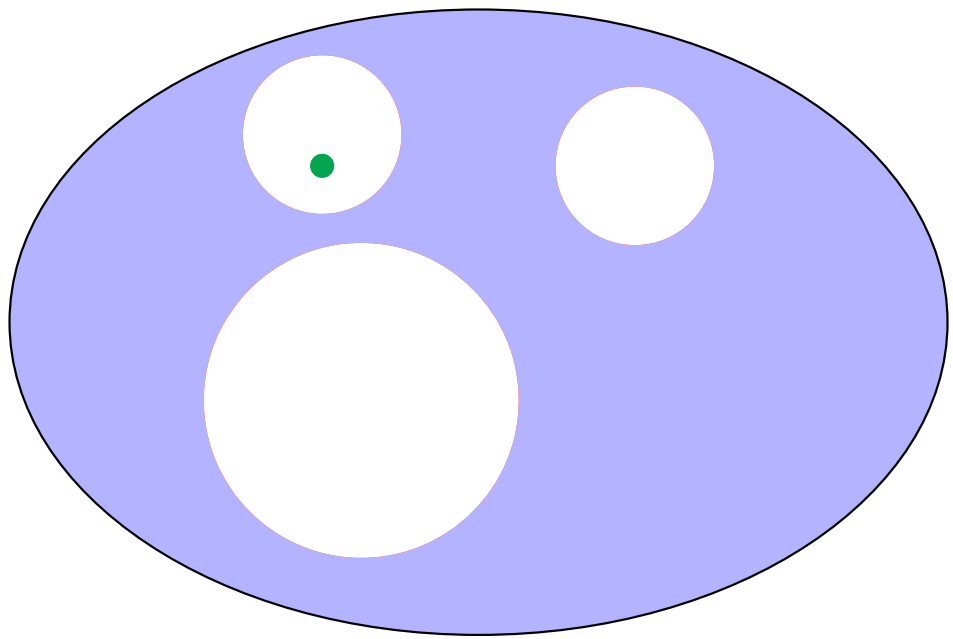

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Generalizing candidates

Key insight

Relaxing variables in a solution can only increase its model counting

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Relaxing variables in a solution can only increase its model counting

Trial and error generalization:

- Try to release one variable from the solution: $x|_{\mathcal{E}'}$
- If $|\phi(x|_{\mathcal{E}'}, \mathcal{Y}, \mathcal{Z})|$ stays below the current maximum: use $x|_{\mathcal{E}'}$ as the generalization from now on.

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Generalizing candidates

Generalization is an instance of a broader problem: *Minimal Subset to a Monotonous Predicate*.

Generic solvers exist:

- QuickXPlain[1]: recursively split the solution
- MergeXPlain[2]: extend QuickXPlain to return multiple solutions

[1] Junker, "QuickXplain: Conflict detection for arbitrary constraint propagation algorithms", 2001

[2] Shchekotykhin, Jannach, and Schmitz, "MergeXplain: Fast computation of multiple conflicts for diagnosis", 2015

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In our case, trial and error seems to be better.

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Theorems

Theorem (Exact #SAT oracle)

BaxMC *is correct, i.e. always returns the actual maximum.*

Theorem (Approximate #SAT oracle)

BaxMC *is probably approximately correct, i.e. returns the correct answer within a margin of* ϵ*, with a probability of* δ*.*
Assuming that the $\#SAT$ *oracle is correct within a margin of* ϵ' *with a probability of* δ'*.*

Related works

MaxCount[1]: amplification then sampling

`d4max`[2]: knowledge compilation to speed-up model counting

`erssat`[3]: more general problem, takes inspiration from *quantified boolean formulas*

[1] Fremont, Rabe, and Seshia, "Maximum model counting", 2017

[2] Audemard, Lagniez, and Miceli, "A New Exact Solver for (Weighted) Max#SAT", 2022

[3] Lee, Wang, and Jiang, "Solving Exist-Random Quantified Stochastic Boolean Satisfiability via Clause Selection", 2018

Some experimental results

Faster for approximate resolution on state-of-the-art benchmarks[1]:

Benchmark name	BaxMC	MaxCount
`backdoor-32-24*`	34.50	231.87
`backdoor-2x16-8*`	60.02	6512.28
`pwd-backdoor`	236.87	TO
`bin-search-16`	1048.43	1490.44
`CVE-2007-2875`	36.14	TO
`CVE-2009-3002`	TO	MO
`reverse`	TO	MO

Yields better results in nearly all cases.

[1] Fremont, Rabe, and Seshia, "Maximum model counting", 2017

Extensions of this algorithm

Implemented optimizations:

- Symmetry breaking
- Equivalent literals handling

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Simple heuristic to prioritize search:

1. When overapproximating the count of a subspace, it may be above the current maximum
2. It may contain promising witnesses
3. Focus the future search in these areas

Future works

On BaxMC resolution:

- Incremental
- Parallel
- Extend to solve stochastic SAT[1] instances

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Actually use this in a security setting.

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Take-home message

- New solver for $\mathrm{Max\#SAT}$
- Two solving modes: exact and approximate
- Better experimental performances

Docker and sources available at:
`https://www-verimag.imag.fr/~vigourth/research/baxmc/`

@VERIMAG (Grenoble):

- Professorship
- Tenured full-time research positions
- Possibilities of post-docs

ask David Monniaux (in audience)

Bootstrapping the search

Idea:

1. Search for a good solution *fast* using a different method
2. Start BaxMC with the values found by the earlier method

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Consequences:

- The result is at least better
- Guarantees about the optimality of the results

References

- Audemard, Gilles, Jean-Marie Lagniez, and Marie Miceli. "A New Exact Solver for (Weighted) Max#SAT". In: *25th International Conference on Theory and Applications of Satisfiability Testing, SAT 2022, August 2-5, 2022, Haifa, Israel*. Ed. by Kuldeep S. Meel and Ofer Strichman. Vol. 236. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022, 28:1–28:20. DOI: 10.4230/LIPIcs.SAT.2022.28. URL: https://doi.org/10.4230/LIPIcs.SAT.2022.28.
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